M208/G
2017 exam revised for refreshed module

Module Examination 2017

Pure Mathematics

Thursday 8 June 2017 **10.00 am – 1.00 pm**

Time allowed: 3 hours

This revised paper is intended to give you an idea of how the June 2017 M208 exam paper might have looked had it been prepared for the refreshed module (first presented in 18J). The old questions have been used as far as possible, but put into the new format for the exam paper with some changes to notation and wording where appropriate. The new paper has only 10 questions for Part 1 instead of 12 and the two questions that have been removed are given after the end of the exam paper.

Question 5(c) and Question 9(b) are new, to justify these questions being awarded 8 marks instead of 6. All of the other 8 mark questions were originally awarded 6 marks but it is appropriate to award them 8 marks in the new exam format.

The rubric, normally on this front page, follows on the next page. The wording of the instructions is as used from June 2020 onwards.

There are **three sections** in this examination.

In **Section 1** you should **attempt all 10 questions**
This section is worth 70% of the total mark.

In **Section 2** you should **attempt 1 out of the 3 questions**. Each question is worth 15% of the total mark.

In **Section 3** you should **attempt 1 out of the 2 questions**. Each question is worth 15% of the total mark.

Include all your working, as some marks are awarded for this.
Write your answers in the answer book(s) provided in **pen**, though you may draw diagrams in pencil.
Start your answer to each question on a new page, clearly indicating the number of the question.
Crossed out work will not be marked.

Calculators are NOT permitted in this examination.

Section 1

You should **attempt all questions**. **This section is worth 70%.**

Question 1 – 5 marks

Sketch the graph of the function f defined by

$$f(x) = \frac{2x+3}{x+1}.$$

Your sketch should identify:

- any asymptotes to the graph;
- any points where the graph crosses the axes. [*5*]

Question 2 – 5 marks

(a) Write down the converse of the following statement for positive integers m and n.

If 5 divides each of m and n, then 5 divides $m + 2n$. [*1*]

(b) Of the statement in part (a) and its converse, one is true and one is false.

Prove the true statement and give a counterexample to the false statement. [*4*]

Linear algebra questions (Book C)

Question 3 – 6 marks

This question concerns the system of linear equations

$$\begin{aligned} x + y + z &= -1 \\ 4x + y - 2z &= 2 \\ 3x - y - 5z &= 5. \end{aligned}$$

(a) Write down the augmented matrix for this system of linear equations. [*1*]

(b) Find the row-reduced form of the matrix that you wrote down in part (a). [*3*]

(c) Use your answer to part (b) to solve the system of linear equations. [*2*]

Question 4 – 6 marks

Let t be the linear transformation

$$\begin{aligned} t : \mathbb{R}^3 &\longrightarrow \mathbb{R}^3 \\ (x, y, z) &\longmapsto (2x + 2z, x - y + 2z, 2y - 2z). \end{aligned}$$

(a) Find a basis for $\operatorname{Im} t$. *[3]*

(b) Determine the dimension of $\operatorname{Ker} t$. *[2]*

(c) State whether t is one-to-one, and justify your answer. *[1]*

Group theory questions (Books B and E)

Question 5 – 8 marks

For each of the following sets, with the binary operation given, determine whether or not it forms a group, justifying your answer.

(a) $(\{1, 2, 4, 8\}, \times_9)$ *[2]*

(b) $(\{1, 3, 9, 11\}, \times_{16})$ *[4]*

(c) $(\{1, 3\}, \times_6)$ *[2]*

Question 6 – 8 marks

The group table for a group G is given below.

	e	a	b	c	d	f	g	h
e	e	a	b	c	d	f	g	h
a	a	b	c	e	h	g	d	f
b	b	c	e	a	f	d	h	g
c	c	e	a	b	g	h	f	d
d	d	g	f	h	e	b	a	c
f	f	h	d	g	b	e	c	a
g	g	f	h	d	c	a	e	b
h	h	d	g	f	a	c	b	e

(a) Show that $H = \{e, b\}$ is a subgroup of G. *[1]*

(b) Write down the left cosets of H in G. *[2]*

(c) Show that H is a normal subgroup of G. *[2]*

(d) Construct the group table of the quotient group G/H, and state a standard group that is isomorphic to G/H. *[3]*

Question 7 – 8 marks

This question concerns the group $(\mathbb{C}^*, \times)$ and the mapping ϕ defined by

$$\begin{aligned} \phi : \mathbb{C}^* &\longrightarrow \mathbb{C}^* \\ z &\longmapsto z^4. \end{aligned}$$

(a) Prove that ϕ is a homomorphism. *[2]*

(b) Determine the kernel and image of ϕ. *[5]*

(c) State a standard group isomorphic to the quotient group $\mathbb{C}^* / \operatorname{Ker} \phi$, justifying your answer briefly. *[1]*

Analysis questions (Books D and F)

Question 8 – 8 marks

Determine whether each of the following series converges or diverges, naming any result or test that you use.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{2n^3+3}{4-3n^3}$ *[3]*

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{3n}{2n^3-n^2+1}$ *[5]*

Question 9 – 8 marks

Determine whether each of the following functions is continuous at 1, naming any result or rule that you use.

(a) $f(x) = \begin{cases} \sin\left(\frac{\pi}{2}x\right), & x \leq 1, \\ \dfrac{2x+1}{x+2}, & x > 1 \end{cases}$ *[5]*

(b) $f(x) = \begin{cases} \sin\left(\frac{\pi}{2}x\right), & x \leq 1, \\ \dfrac{2x-1}{x+2}, & x > 1 \end{cases}$ *[3]*

Question 10 – 8 marks

(a) Prove that

$$\int_0^{\pi/2} \left(x^3 + \cos\left(x^2\right)\right) dx \leq \frac{\pi^4}{64} + \frac{\pi}{2}.$$ *[5]*

(b) Use Stirling's Formula to prove that

$$\frac{(4n)!}{4^{4n}(n!)} \sim 2\left(\frac{n}{e}\right)^{3n} \quad \text{as } n \to \infty.$$ *[3]*

Section 2

You should attempt **one question**. If you attempt more, the score from your best question will count.

Each question is worth 15%.

Question 11

This question concerns the matrix

$$\mathbf{A} = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}.$$

(a) Show that $(2, -1, -1)$ is an eigenvector of $\mathbf{A}$, and find the corresponding eigenvalue. *[2]*

(b) Use the characteristic equation of $\mathbf{A}$ to check the eigenvalue that you obtained in part (a), and find the remaining eigenvalues of $\mathbf{A}$. *[3]*

(c) Find the eigenspaces of $\mathbf{A}$. *[5]*

(d) Find an orthonormal eigenvector basis of $\mathbf{A}$. *[2]*

(e) Write down an orthogonal matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{P}^T\mathbf{A}\mathbf{P} = \mathbf{D}$. *[3]*

Question 12

The permutations $p = (1\,2\,6)(3\,4)$ and $q = (2\,5\,3\,6)$ are elements of S_6.

(a) (i) Find each of the following as a permutation in cycle form:

$$p^{-1}, \quad q^2, \quad q \circ p, \quad q \circ p \circ q^{-1}.$$

(ii) State the order of each of p, q, q^2 and $q \circ p$.

(iii) Write each of p, q and $q \circ p$ as a composite of transpositions, and hence determine the parity of each of p, q and $q \circ p$. *[7]*

(b) (i) Determine the subgroup H of S_6 generated by p, giving the elements of H in cycle form.

(ii) Explain why $s = (1\,4\,3\,5)$ is conjugate to q in S_6 and determine all the elements of S_6 which conjugate s to q. *[8]*

Question 13

(a) Determine whether each of the following sequences (a_n) converges or diverges, naming any result or rule that you use. If a sequence does converge, then find its limit.

(i) $a_n = \dfrac{2n^4 + 3^n}{n^2 + 4(3^n)}$

(ii) $a_n = \dfrac{n^3 + 3^n}{2n^2 + (-3)^n}$

(iii) $a_n = \dfrac{3^n + n!}{2n^4 + 4^n}$ *[11]*

(b) Determine the least upper bound of the set

$$E = \left\{ 3 - \frac{2}{n^2} : n = 1, 2, \ldots \right\}.$$ *[4]*

Section 3

You should attempt **one question**. If you attempt more, the score from your better question will count.

Each question is worth 15%.

Question 14

The set of matrices

$$G = \left\{ \begin{pmatrix} 1 & 0 \\ b & a \end{pmatrix} : a, b \in \mathbb{R},\ a \neq 0 \right\}$$

is a group under matrix multiplication.

(a) Show that the following equation defines a group action of G on the plane $\mathbb{R}^2$:

$$\begin{pmatrix} 1 & 0 \\ b & a \end{pmatrix} \wedge (x, y) = (x, bx + ay).$$ [4]

The remainder of this question refers to this group action.

(b) (i) Find the orbit of each of

$$(1, 0);\ (0, 1);\ (-2, 1).$$

(ii) Give a geometric description of all the orbits of the action. [7]

(c) Find the stabiliser of each of

$$(1, 0);\ (-2, 1).$$ [2]

(d) Find $\operatorname{Fix} \begin{pmatrix} 1 & 0 \\ 2 & 2 \end{pmatrix}$. [2]

Question 15

(a) Determine the Taylor polynomial $T_2(x)$ at -1 for the function

$$f(x) = \frac{x-1}{x-3}.$$

Show that $T_2(x)$ approximates $f(x)$ with an error less than 0.001 on the interval $[-1.5, -1]$. [8]

(b) (i) Use the $\varepsilon - \delta$ definition of continuity to prove that the function

$$f(x) = x^3 - x^2$$

is continuous at 1.

(ii) State whether the function

$$f(x) = \frac{\sin x + \cos x}{4 - \sin x}$$

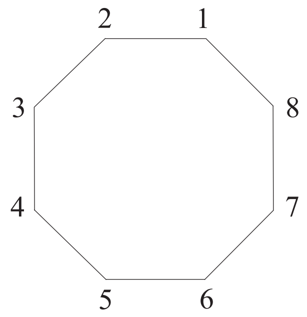
is uniformly continuous on the interval $[0, 2\pi]$, and justify your answer. [7]

[**END OF QUESTION PAPER**]

The following are questions from the 2017 paper that could alternatively have been included.

Alternative Question 6 – 8 marks

This question concerns the symmetry group G of the regular octagon shown below.



Let g be the rotation through an angle of $\pi/2$ anticlockwise about the centre of the octagon, and let h be the reflection in the line through the vertices at locations 2 and 6.

(a) Write g, g^2 and h in cycle form, using the labelling of the vertex locations shown above. [*3*]

(b) Express the conjugate $g \circ h \circ g^{-1}$ of h by g in cycle form and identify this conjugate as a symmetry of the octagon. [*2*]

(c) Write down the conjugacy class of G that contains h. [*3*]

Alternative Question 10 – 8 marks

A question similar to this could have been included, with a limit requiring two applications of l'Hôpital's Rule instead of one, in order to make the question worth 8 marks.

Prove that the following limit exists, and determine its value.

$$\lim_{x\to 0} \frac{\sin 2x + \cos 2x - 1}{e^{-2x} + x - 1}$$ [*6*]

[END OF QUESTION PAPER]